

Dirac operator normality and chiral fermions ^{*}

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Normality of the Dirac operator is shown to be necessary for chiral properties. From the global chiral Ward identity, which in the continuum limit gives the index theorem, a sum rule results which constrains the spectrum. The Ginsparg-Wilson relation is to be restricted to its simple form and is a member of a set of spectral constraints. A family of alternative chiral transformations is introduced. The one of Lüscher is a special case which transports only the anomaly term to the measure. An alternative transformation would also be needed to correct Fujikawa's path-integral approach. From a general function of the hermitean Wilson-Dirac operator the one of Neuberger follows.

I. Introduction

Imposing the Ginsparg-Wilson (GW) relation [1] in Refs. [2, 3] a particular formula for the index of the massless Dirac operator D on the lattice has been given. Chiu [4] has observed that with the simple form of the GW relation and γ_5 -hermiticity the Dirac operator D gets normal and its index and the corresponding difference at its second real eigenvalue add up to zero. This raises the questions what precisely the general conditions for chiral properties are and which general rules follow.

Assuming the GW relation, Lüscher [3] has introduced an alternative chiral transformation under which the measure is no longer invariant, a generalized finite form of which has been given by Chiu [5]. This reminds of an old claim by Fujikawa [6] in continuum theory that the chiral anomaly could be obtained from the measure. Thus it appears appropriate now to clarify in general what the rôle of the integration measure is.

Neuberger [7] has derived an explicit form of the massless Dirac operator on the lattice from the overlap formalism [8]. It is of interest whether this form, which relies on

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the hermitean Wilson-Dirac operator H , also follows from other general requirements and under which conditions it is the only solution of the GW relation.

In the following we start from the general Ward identity holding in a background gauge field. In this context we also introduce a family of alternative chiral transformations which give the same Ward identity, however, allow to transport terms between the action contribution and the measure contribution.

Next we show that it is necessary, in addition to γ_5 -hermiticity, to have normality of the Dirac operator D in order to get chiral eigenstates and thus chiral properties. We derive general consequences for the terms in the Ward identity, which then gets the general sum rule for chiral differences. This rule is seen to put severe restrictions on the spectrum of D which are crucial for allowing a nonvanishing index.

With respect to the GW relation we notice that its general form does not guarantee normality of D so that we have to restrict to its simple form. We observe that, given γ_5 -hermiticity, this relation is actually a spectral constraint. Using a decomposition of D we give an example of a family of spectral constraints of which the GW one is a member. The alternative transformation which transports the anomaly term to the measure in the GW case is seen to get that of Lüscher.

We point out that in the continuum limit the index theorem follows from the lattice Ward identity and one has still $\text{Tr}(\gamma_5) = 0$. With different origins of the anomaly term, there is agreement with the expectations of conventional continuum theory. However, the derivation of Fujikawa's path-integral approach turns out not to agree with what is well defined from the lattice. A correction of this approach would include the use of an alternative transformation.

To study what follows starting from the hermitean Wilson-Dirac operator H we require D to be normal, γ_5 -hermitean, and a general function of H . This is seen to lead to the operator of Neuberger. It also establishes the connection to the GW relation.

II. Ward identities

Fermionic Ward identities arise from the condition that $\int [d\bar{\psi}d\psi]e^{-S_f}\mathcal{O}$ must not change under a transformation of the integration variables. Considering in particular the transformation $\psi' = \exp(i\eta\Gamma)\psi$, $\bar{\psi}' = \bar{\psi}\exp(i\eta\bar{\Gamma})$ this can be expressed by the identity

$$\frac{d}{d\eta} \int [d\bar{\psi}'d\psi']e^{-S'_f}\mathcal{O}' \Big|_{\eta=0} = 0 \quad (1)$$

where $S'_f = \bar{\psi}' M \psi'$. Evaluation of (1) gives

$$i \int [d\bar{\psi} d\psi] e^{-S_f} \left(-\text{Tr}(\bar{\Gamma} + \Gamma) \mathcal{O} - \bar{\psi}(\bar{\Gamma} M + M \Gamma) \psi \mathcal{O} + \bar{\psi} \bar{\Gamma} \frac{\partial \mathcal{O}}{\partial \bar{\psi}} - \frac{\partial \mathcal{O}}{\partial \psi} \Gamma \psi \right) = 0, \quad (2)$$

with three contributions, one from the derivative of the integration measure, one from that of the action, and one from that of \mathcal{O} .

In the present context one usually puts $\mathcal{O} = 1$. We can, however, do better integrating out the ψ and $\bar{\psi}$ fields in the second term of (2), which after a calculation relying on Grassmann properties gives

$$i \text{Tr} \left(-\bar{\Gamma} - \Gamma + M^{-1}(\bar{\Gamma} M + M \Gamma) \right) \int [d\bar{\psi} d\psi] e^{-S_f} \mathcal{O} = 0. \quad (3)$$

From (3) it becomes obvious that in a background gauge field the expectation value factorizes so that also for arbitrary \mathcal{O} (and not only for $\mathcal{O} = 1$) it suffices to consider the identity

$$\frac{1}{2} \text{Tr} \left(-\bar{\Gamma} - \Gamma + M^{-1}(\bar{\Gamma} M + M \Gamma) \right) = 0 \quad (4)$$

where $-\frac{1}{2} \text{Tr}(\bar{\Gamma} + \Gamma)$ is the measure contribution and $\frac{1}{2} \text{Tr}(M^{-1}(\bar{\Gamma} M + M \Gamma))$ the action contribution.

For the global chiral transformation, in which case one has $\Gamma = \bar{\Gamma} = \gamma_5$, the measure contribution vanishes and (4) becomes

$$\frac{1}{2} \text{Tr}(M^{-1} \{\gamma_5, M\}) = 0. \quad (5)$$

Obviously this can also be read as $\text{Tr}(\gamma_5) = 0$, of which the Ward identity is the particular decomposition which is dictated by the chiral transformation.

In the presence of zero modes of a Dirac operator D it is crucial to take care that the derived relations remain properly defined. To guarantee this we put $M = D + \varepsilon$ and let ε go to zero in the final result. Thus from (5) we altogether have

$$\text{Tr}(\gamma_5) = \frac{1}{2} \text{Tr}((D + \varepsilon)^{-1} \{\gamma_5, D\}) + \varepsilon \text{Tr}((D + \varepsilon)^{-1} \gamma_5) = 0 \quad (6)$$

with γ_5 , of course, understood as γ_5 times the appropriate unit operator. To have definite names in our discussions we shall call the first term in (6), which only contributes if $\{\gamma_5, D\} \neq 0$, anomaly term, and the second one mass term or index term (for $\varepsilon \rightarrow 0$). The operator D is considered to be massless.

We note that a family of alternative global chiral transformations can be introduced putting $\Gamma = \gamma_5 - K$, $\bar{\Gamma} = \gamma_5 - \bar{K}$, which inserted into (4) gives

$$-\frac{1}{2}\text{Tr}(\bar{\Gamma} + \Gamma) = +\frac{1}{2}\text{Tr}(K + \bar{K}) \quad (7)$$

for the measure contribution and

$$\frac{1}{2}\text{Tr}(M^{-1}(\bar{\Gamma}M + M\Gamma)) = \frac{1}{2}\text{Tr}(M^{-1}\{\gamma_5, M\}) - \frac{1}{2}\text{Tr}(K + \bar{K}) \quad (8)$$

for the action contribution. Obviously the extra term of the latter cancels the measure term so that again the result (5) is obtained for any operators K and \bar{K} .

While the Ward identity remains the same for these transformations, they may be used to change the origin of its terms. For example, with

$$K = \frac{1}{2}M^{-1}\{\gamma_5, D\} \quad , \quad \bar{K} = \frac{1}{2}\{\gamma_5, D\}M^{-1} \quad (9)$$

the anomaly term of (6) is transported from the action contribution to the measure contribution.

To get the local chiral transformations one simply has to replace γ_5 of the global cases by $\gamma_5\hat{e}(n)$, where $\hat{e}(n)$ in lattice-space representation reads $(\hat{e}(n))_{n''n'} = \delta_{n''n}\delta_{nn'}$. Thus to see the essential features it will suffice to consider the relations of the global case in the following.

III. Chiral properties

The derivation of the identity (4) implies that the occurring operators, acting on a vector space (with dimension number of sites times spinor dimension times gauge-group dimension) map to this space itself. In fact, instead of the Grassmann integrals one can equivalently consider minors and determinants, or generalizations thereof [9], for which this is a prerequisite. The definition of adjoint operators in addition needs an inner product so that the vector space must be a unitary one. This then allows to define normal operators (and their special cases as e.g. hermitean and unitary ones) and is also necessary for using the notion of γ_5 -hermiticity.

We require D to be normal and γ_5 -hermitean which will be seen in the following to be necessary for really having chiral properties. By normality, $[D, D^\dagger] = 0$, one gets simultaneous eigenvectors of D and D^\dagger . Then with the eigenequation

$$Df_k = \lambda_k f_k \quad (10)$$

we also have

$$D^\dagger f_k = \lambda_k^* f_k \quad (11)$$

where, to get the eigenvalue, also the inner product has been used. From (11) by γ_5 -hermiticity, $D^\dagger = \gamma_5 D \gamma_5$, we obtain the equation

$$D \gamma_5 f_k = \lambda_k^* \gamma_5 f_k \quad (12)$$

which has important consequences. The comparison of (10) multiplied by γ_5 with (12) gives

$$[\gamma_5, D] f_k = 0 \quad \text{if} \quad \lambda_k \text{ real} \quad (13)$$

so that in the subspace of real eigenvalues of D one can introduce simultaneous eigenvectors of D and of γ_5

$$\gamma_5 f_k = c_k f_k \quad \text{for} \quad \lambda_k \text{ real} \quad (14)$$

with the chirality c_r taking values $+1$ and -1 . The comparison of (10) with (12) shows that one has simultaneously

$$D f_k = \lambda_k f_k \quad \text{and} \quad D \gamma_5 f_k = \lambda_k^* \gamma_5 f_k \quad \text{for} \quad \lambda_k \neq \lambda_k^*, \quad (15)$$

i.e. pairs of eigenvectors related to conjugate complex eigenvalues.

Conversely, given γ_5 -hermiticity, (12) implies normality of D . Further, having the chiral subspace, (12) follows for real eigenvalues. For complex ones it follows from (15). Thus to have (13) and (15) normality of D is also necessary.

Normality thus turns out to be necessary to get the chiral subspace which is the basis of chiral properties of D . In addition normality of D , being necessary and sufficient in order that its eigenvectors form a complete orthonormal set, guarantees the completeness of eigenvectors which in the following will be seen to be crucial for the index relations.

We note that if one would try to do without (15), one would have normality of D only in the subspace of real eigenvalues. To specify D generally in such a way appears not feasible. Further, in the subspace of complex eigenvalues completeness of the eigenvectors then would only be guaranteed if there were no degeneracies of eigenvalues [10]. To specify D generally such that the respective nondiagonal cases are excluded appears again not feasible. Thus we must insist that D be normal in all unitary space.

Multiplying (10) from the left by $f_l^\dagger \gamma_5$ and its adjoint $f_l^\dagger D^\dagger = f_l^\dagger \lambda_l^*$ from the right by $\gamma_5 f_k$, and using γ_5 -hermiticity, one obtains the relation

$$f_l^\dagger \gamma_5 f_k = 0 \quad \text{for} \quad \lambda_l^* \neq \lambda_k. \quad (16)$$

We note that if one makes use of the properties given by (14) and (15) the relation (16) reflects the orthogonality of eigenvectors with different eigenvalues. This is most easily seen introducing for the eigenvectors f_k the more detailed notations $f_k^{(5)}$ for $\text{Im}\lambda_k = 0$, $f_k^{(1)}$ for $\text{Im}\lambda_k > 0$, and $f_k^{(2)} = \gamma_5 f_k^{(1)}$ for $\text{Im}\lambda_k < 0$.

With (14), (16), and the completeness of the eigenvectors of D we obtain for the terms in the identity (6) and for this identity itself

$$\lim_{\varepsilon \rightarrow 0} \text{Tr}((D + \varepsilon)^{-1} \gamma_5 \varepsilon) = N_+(0) - N_-(0) \quad (17)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} \text{Tr}((D + \varepsilon)^{-1} \{\gamma_5, D\}) = \sum_{\lambda \neq 0 \text{ real}} (N_+(\lambda) - N_-(\lambda)) \quad (18)$$

$$\text{Tr}(\gamma_5) = \sum_{\lambda \text{ real}} (N_+(\lambda) - N_-(\lambda)) = 0 \quad (19)$$

where the numbers of modes with chirality ± 1 at a real eigenvalue λ of D are given by $N_{\pm}(\lambda) = \sum_k (\lambda_k = \lambda \text{ real}) (1 \pm c_k)/2$.

It is seen that (17) gives the index $N_-(0) - N_+(0)$ of D . The r.h.s. of (18) exhibits a form characteristic of the anomaly term. The sum rule for real modes (19) shows that one has the same total number of right-handed and of left-handed modes. The mechanism leading to a nonvanishing index thus is seen to work via compensating numbers of modes at different λ .

From (19) it follows that the index of D can only be nonvanishing if a corresponding difference from nonzero eigenvalues exists. This requires that in addition to 0, allowing for zero modes, there must be at least one further real value available in the spectrum in order that the index can be nontrivial. Thus it turns out that this sum rule puts severe restrictions on the spectrum of D . Obviously it is a novel manifestation of the fact that a nontrivial index requires breaking of the chiral symmetry.

IV. Remarks on GW relation

From the general GW relation [1] $\{\gamma_5, D\} = D\gamma_5 R D$, using γ_5 -hermiticity of D and $[\gamma_5, R] = 0$, one obtains $[D, D^\dagger] = 2D^\dagger[R, D]D^\dagger$. Therefore one should have $[R, D] = 0$ in order that D gets normal which, as we have seen, is crucial for chiral properties and their consequences in gauge theories. Because it is necessary to satisfy the relation $[R, D] = 0$ in a general way this means to put R equal to a multiple of the identity.

Thus, having to insist on normality of D , we remain with the simple form of the GW relation

$$\{\gamma_5, D\} = \rho^{-1} D \gamma_5 D \quad (20)$$

with ρ being a real constant. Requiring also γ_5 -hermiticity of D , the condition (20) means that $\rho(D + D^\dagger) = DD^\dagger = D^\dagger D$ should hold, i.e. that $D/\rho - 1$ should be unitary. Thus the actual content of (20) is the restriction of the spectrum of D to the circle through zero with center at ρ . The crucial properties then are that real eigenvalues get possible at 0 and at 2ρ , allowing for zero modes and for a nonzero index, respectively.

Imposing the GW relation (20), in the massless case the anomaly term in (6) can be expressed as

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} \text{Tr} \left((D + \varepsilon)^{-1} \{\gamma_5, D\} \right) = (2\rho)^{-1} \text{Tr}(\gamma_5 D) \quad (21)$$

so that the identity (6) can be replaced by

$$\text{Tr}(\gamma_5) = (2\rho)^{-1} \text{Tr}(\gamma_5 D) + \lim_{\varepsilon \rightarrow 0} \text{Tr} \left((D + \varepsilon)^{-1} \gamma_5 \varepsilon \right) = 0. \quad (22)$$

The relation $\text{Tr}(\gamma_5 D) = \sum_{\lambda \neq 0 \text{ real}} \lambda (N_+(\lambda) - N_-(\lambda))$, which would be not useful in the general case, now simplifies to $\text{Tr}(\gamma_5 D) = 2\rho (N_+(2\rho) - N_-(2\rho))$ and the sum rule (19) to $\text{Tr}(\gamma_5) = N_+(0) - N_-(0) + N_+(2\rho) - N_-(2\rho) = 0$. The combination of these relations is what gives the formula

$$(2\rho)^{-1} \text{Tr}(\gamma_5 D) = N_-(0) - N_+(0) \quad (23)$$

considered in [2, 3] for the index.

Using (20), one can replace (9) of the alternative transformation transporting the anomaly term by $K = (2\rho)^{-1} \gamma_5 D$, $\bar{K} = (2\rho)^{-1} D \gamma_5$. This obviously gives the transformation introduced in the GW case by Lüscher [3], tailored to make the classical action $\bar{\psi} D \psi$ invariant. The measure contribution (7) then gets $(2\rho)^{-1} \text{Tr}(\gamma_5 D)$. However, there still remains the action contribution $\lim_{\varepsilon \rightarrow 0} \text{Tr} \left((D + \varepsilon)^{-1} \gamma_5 \varepsilon \right)$.

The remaining action contribution is missing in [3] since no zero-mode regularization has been used. Thus it looks there like the action would also be invariant in the quantum case with zero modes, as is not correct. In a separate next step, which implicitly uses the decomposition (22) of $\text{Tr}(\gamma_5) = 0$, what should have been obtained from the action contribution is calculated from the measure term. This does not cure the missing in the originally derived identity.

Clearly one can think of many possibilities satisfying the requirement that in the spectrum one should allow for at least one further real value in addition to 0, as imposed by the sum rule (19). For finding appropriate constraints the decomposition

$$D = u + iv \quad \text{with} \quad u = u^\dagger = \frac{1}{2}(D + D^\dagger) \quad , \quad v = v^\dagger = \frac{1}{2i}(D - D^\dagger) \quad (24)$$

appears useful. The reason for this is that by normality of D one obtains $[u, v] = 0$ so that for u , v , and D one gets simultaneous eigenvectors and the eigenvalues of u and v are simply the real and imaginary parts, respectively, of those of D .

For example, one may use this to define a family of constraints to a one-dimensional set, allowing eigenvalues at zero and at one further real value, by

$$v^2 = 2\rho u + (\beta - 1)u^2 \quad \text{with} \quad \beta \geq 0, \beta \neq 1 \quad (25)$$

in which case the spectrum is restricted for $\beta = 0$ to the circle of the GW case, for $0 < \beta < 1$ to ellipses, and for $1 < \beta$ to hyperbolas. Inserting u and v and using γ_5 -hermiticity this may be cast into the form

$$\{\gamma_5, D\} = \rho^{-1} \left(\left(1 - \frac{\beta}{2}\right) D \gamma_5 D - \frac{\beta}{4} \{\gamma_5, D^2\} \right) \quad (26)$$

which generalizes the relation (20).

V. Continuum limit

For the present purpose it suffices to consider the continuum limit to the quantum field theory of fermions in a background gauge field. The limit of the anomaly term in the identity (6) has been shown to be

$$\frac{1}{2} \text{Tr} \left((D + \varepsilon)^{-1} \{\gamma_5, D\} \right) \rightarrow - \frac{g^2}{32\pi^2} \int d^4x \text{tr}(\tilde{F}F) \quad (27)$$

long ago [11] for the Wilson-Dirac operator and recently [12, 13] also for the operator of Neuberger [7]. In the latter case one has to note that the l.h.s. of (27) can be replaced according to (21) to get the form used in [12, 13]. Though there are still subtleties [12] which deserve further development, it can be expected that any appropriate form of D should give (27).

In the massless case with normal and γ_5 -hermitean D we can insert (17) and (27) into the identity (6) to obtain

$$\text{Tr}(\gamma_5) = - \frac{g^2}{32\pi^2} \int d^4x \text{tr}(\tilde{F}F) + N_+(0) - N_-(0) = 0 . \quad (28)$$

Thus obviously the index theorem follows in the limit. To see that one also still has $\text{Tr}(\gamma_5) = 0$ one has to note that any complete set of vectors can be used to calculate $\text{Tr}(\gamma_5)$. In particular, one may select a set which exploits the fact that the spinor space factorizes off. Since in the latter space one has $\text{tr}(\gamma_5) = 0$, the sequence for $\text{Tr}(\gamma_5)$ with increasing lattice spacing is one with all members zero, so that one has indeed $\text{Tr}(\gamma_5) = 0$ also in the limit.

We emphasize that, quite remarkably, the index theorem follows here in a rather different setting from that of mathematics. There the Atiyah-Singer theorem is obtained solely considering the continuum Dirac (or Weyl) operator on a compact manifold finding that its index equals a topological invariant. Here we consider the nonperturbative formulation of the quantum field theory of fermions in a background gauge field and derive the chiral Ward identity. This identity then gives the index theorem. An essential property of this theory is that a chirally noninvariant modification occurs in its action. Additional features of the field-theoretic setting are that the Ward identity is a particular decomposition of $\text{Tr}(\gamma_5) = 0$ and that one gets a local version explaining the nonconservation of the singlet axial-vector current.

We now compare with the conventional continuum approach, in which (in our notation) the operator D is antihermitean (and thus also normal) and γ_5 -hermitean. Because one then has $\{\gamma_5, D\} = 0$, the anomaly term in the identity (6) vanishes. However, at the level of the Ward identity in perturbation theory (in the well known triangle diagram) one gets an ambiguity which, if fixed in a gauge-invariant way, produces the anomaly term [14]. Thus, though with different origin of this term, there one gets agreement with (28).

Nevertheless, there is an essential difference. While in the continuum the chirally noninvariant modification of the theory occurs only at the level of the Ward identity, on the lattice the origin of the anomaly sits in the action itself. Thus, since deriving things from the start is more satisfactory than only fixing inconsistencies later by hand, the lattice formulation is the preferable one. The missing of an appropriate modification at the level of the action in the continuum approach has the concrete consequence that there are difficulties with making it truly nonperturbative.

This is seen noting that the respective attempts rely on the Pauli-Villars (PV) term. The motivation there is that in perturbation theory in the PV difference ambiguous contributions, being mass-independent, drop out so that the PV term gives the anomaly [14]. Assuming the PV term to be nonperturbatively valid the desired result is obtained neglecting higher orders in the PV mass [15]. However, one actually gets zero as one readily checks using $\text{Tr}(\gamma_5) = 0$, the neglect of the sum of higher orders being not correct. This does not come as a surprise since in the lattice formulation it is obvious that a chirally noninvariant modification of the action is indispensable to get the correct result.

In the path-integral approach [6] the usual chiral transformation is used so that in the global case the measure contribution is $-\text{Tr}(\gamma_5)$. Arguing that it should be regularized this contribution, which is actually zero, is replaced by a term which can be checked to be equivalent to the PV term in [15] and from which essentially as in [15], i.e. incorrectly as pointed out above, the anomaly is obtained. On the other hand, in the Ward identity then to the mass (index) term in the action contribution the anomaly term is not added, as would have been necessary in the continuum [14]. This compensates the unjustified replacement of $\text{Tr}(\gamma_5)$ so that the desired result is obtained.

From our results it is obvious that to correct the procedure of [6] one firstly has to use the alternative transformation with (9) which transports the anomaly term to the measure and secondly to take care that this term emerges properly in a nonperturbative way (requiring an appropriate modification of the action as is e.g. provided by Wilson's regularization suppressing doublers in lattice theory). It should be added that the defects of the approach in [6] also invalidate recent lattice considerations [16] which rely on it.

VI. Normal D from hermitean H

To get an explicit form of D one can start from the Wilson-Dirac operator or some generalization of it, which is γ_5 -hermitean, however, not normal. The γ_5 -hermiticity of this operator X implies that $H = \gamma_5 X$ is even hermitean so that its spectral representation allows to define functions of H . This suggests to get a normal operator D from a general function of H by imposing the necessary conditions.

To proceed it is convenient to consider $F = \gamma_5 D$ which should generalize $H = \gamma_5 X$. From γ_5 -hermiticity of D it follows that F must be hermitean and normality of D gives the condition $[\gamma_5, F^2] = 0$. This does, however, not yet determine F . In fact, with $[\gamma_5, E^2] = 0$ for some hermitean operator E the conditions on F are satisfied by $F^2 = E^2 + \{\gamma_5, Y\} + c$ where Y is some hermitean operator and c some real number. From the fact that F^2 is a square it then follows that one must have $c = b^2$ nonnegative and $Y = bE$. One thus arrives at $F = E + b\gamma_5$ in which the operator E and the real number b are to be determined.

With $H\phi_l = \alpha_l\phi_l$ the definition of E as a function of H is $E(H) = \sum_l E(\alpha_l)\phi_l\phi_l^\dagger$ where $E(\alpha)$ is a real function of real α . The task then is to determine $E(\alpha)$ in such a way that the condition $[\gamma_5, E(H)^2] = 0$ holds. Because H does not commute with γ_5 and since we are not allowed to restrict H in any way this can only be achieved by requiring the function $E(\alpha)^2$ in $E(H)^2 = \sum_l E(\alpha_l)^2\phi_l\phi_l^\dagger$ to be constant. Thus we get $E(H)^2 = \rho^2\mathbb{1}$ and $E(\alpha) = \pm\rho$ with ρ being a real constant.

From $E(H)^2 = \rho^2\mathbb{1}$ we see that $\gamma_5 E(H)/\rho$ is unitary so that the spectrum of $\gamma_5 E(H)$

is on a circle with radius $|\rho|$ and center at zero. To allow for zero modes of D we therefore in $D = \gamma_5 F = \gamma_5 E + b$ have to choose $b = \rho$ or $b = -\rho$ and cannot admit any dependence of b on H . Without restricting generality taking $b = \rho$ we thus get $D = \rho(1 + \gamma_5 \epsilon_0(H))$ where $\epsilon_0(\alpha) = \pm 1$. For this form of D obviously already the GW relation (20) holds.

For $\alpha \neq 0$, requiring the function $\epsilon_0(\alpha)$ to be odd and nondecreasing, it gets the sign function defined by $\epsilon(\alpha) = \pm 1$ for $\alpha \gtrless 0$. That this choice is appropriate is confirmed by checking the classical continuum limit in the free case. Thus if all $\alpha_l \neq 0$ we obtain

$$D = \rho \left(1 + \gamma_5 \epsilon(H) \right) \quad (29)$$

which is seen to be just the operator of Neuberger [7].

If $\alpha_l = 0$ occur also $\epsilon(0)$ is to be specified. Because of the condition $E(H)^2 = \rho^2 \mathbb{1}$ only either $+1$ or -1 is possible for this. To prefer none of these choices we propose to calculate (29) independently for each choice of $\epsilon(0)$ and to take the mean of the final results. As will be seen below this gives agreement with what follows from counting eigenvalue flows of H .

To the get index of D by counting eigenvalue flows of H has been introduced in [8]. These flows with m rely on the form $H(m) = H(0) + m\gamma_5$ of the hermitean Wilson-Dirac operator and, with the eigenequation $H(m)\phi_l(m) = \alpha_l(m)\phi_l(m)$, are described by the functions $\alpha_l(m)$. We have recently shown [17] that these spectral flows obey a differential equation and have given a detailed overview of the solutions of this equation.

The relation to the index can be obtained inserting (29) into (23) which in the absence of zero modes of H gives $N_-(0) - N_+(0) = \frac{1}{2}\text{Tr}(\epsilon(H))$ and in terms of numbers of positive and negative eigenvalues of H reads $N_-(0) - N_+(0) = \frac{1}{2}(N_+^H - N_-^H)$. We now note that this form is also adequate in the presence of zero modes of H . In fact, following a flow, up to crossing there is a change by $\frac{1}{2}$ and after this a further change by $\frac{1}{2}$. At the very moment of crossing a change of $\frac{1}{2}$ is reached which obviously agrees with the respective result of the procedure of dealing with $\epsilon(0)$ proposed above.

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